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On soliton excitations in a one-dimensional Heisenberg ferromagnet

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Abstract. Soliton excitations in a one-dimensional Heisenberg ferromagnet are studied by means of the Holstein–Primakoff representation. Writing the Hamiltonian and the equation of motion into dimensionless forms, we show that the relative ratio of ε to η is important for the determination of the modified terms of the non-linear Schrödinger equation; $\varepsilon = 1/\sqrt{S}$ (S is the spin length) is the small dimensionless parameter used in the semiclassical approximation and $\eta = a/\lambda_0$ (a is the lattice space and λ_0 is the characterisic wavelength of the excitations) is another small dimensionless parameter used in the long-wave approximation. The soliton solutions are given in three cases ($\eta = O(\varepsilon)$, $\eta = O(\varepsilon^{3/2})$ and $\eta = O(\varepsilon^2)$) which correspond to the different physical conditions of the system. The results obtained by Pushkarov and Pushkarov, de Azevedo *et al* and Škrinjar *et al* are included in our approach.

1. Introduction

In recent years, there has been considerable interest shown in the study of non-linear excitations in a one-dimensional (1D) Heisenberg ferromagnet. Using the classical approach, Tjon and Wright (1977) obtained single-soliton solutions in the continuum limit of the Hamiltonian. With use of a gauge transformation, Zakharov and Takhtadzhyan (1979) have proved the gauge equivalence of the Heisenberg chain and the non-linear Schrödinger system. In the semiclassical approach of Pushkarov and Pushkarov (1977), the non-linear Schrödinger equation was obtained from the expansion of the Holstein–Primakoff (HP) (1940) transformation. Since the single-soliton solutions are different from those found using classical approach, de Azevedo *et al* (1982) argued that the non-linear terms in the calculations of Pushkarov and Pushkarov were not taken into account properly in obtaining the equation of motion. They gave a 'consistent' treatment and derived a modified non-linear Schrödinger equation. The anisotropic Heisenberg chain was discussed in a similar way by Škrinjar *et al* (1987). Soliton excitations in the anisotropic case in the classical limit have also been investigated (Škrinjar *et al* 1989).

All studies made so far on soliton excitations in a 1D Heisenberg ferromagnet in the HP representation of spin operators are based on two approximations: one is the semiclassical approximation which takes $1/\sqrt{S}$ (S is the spin length) as the small parameter and the other is the long-wave limit in which the coherent amplitudes $\alpha_{j\pm1}$ are expanded to a^2-terms (a is the lattice constant). The two approximations are thought to be independent of each other. Since two small parameters arise in the perturbation expansion, we think that the approach must be used carefully. The aim of this paper is to show that the relative ratio of ε to η is important for the determination of the modified terms of the non-linear Schrödinger equation; $\varepsilon = 1/\sqrt{S}$ is the dimensionless small parameter used in the semiclassical approximation and $\eta = a/\lambda_0$ (λ_0 is the typical wavelength of the waves) is another dimensionless small parameter used in the long-wave limit. The relation between ε and η, from our point of view, should be determined by the corresponding physical conditions of the system. The soliton solutions are discussed in three cases: $\eta = O(\varepsilon)$, $\eta = O(\varepsilon^{3/2})$ and $\eta = O(\varepsilon^2)$. The results obtained by Pushkarov and Pushkarov, de Azevedo *et al* and Škrinjar *et al* can be included in our approach. A comparison between the theory of non-linear long waves in shallow water and our theory is given in section 5.

2. The model Hamiltonian, the HP representation and the semiclassical approximation

We consider an anisotropic Heisenberg chain in an external field f, whose Hamiltonian can be put into the form

$$H - H_0 = -\mu f \sum_i (S_i^z - S\hbar) - \frac{J}{2}\sum_{i,\delta}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+\delta} - S^2\hbar^2) - \frac{J}{2}\tau \sum_{i,\delta}(S_i^z S_{i+\delta}^z - S^2\hbar^2) \qquad (1)$$

where $\boldsymbol{S}_i$ denotes the spin of the ith ion, J is the exchange integral, τ the dimensionless anisotropic parameter, μ the magnetic moment, $\hbar$ the Planck constant divided by 2π and $\delta = \pm 1$. If we define $\tilde{\boldsymbol{S}}_i = \boldsymbol{S}_i/\hbar$ and $\tilde S_i^{\pm} = \tilde S_i^x \pm \mathrm{i}\tilde S_i^y$, equation (1) can be written in the dimensionless form

$$\tilde H = \frac{H - H_0}{JS^2\hbar^2} = -\tilde f \sum_i \frac{\tilde S_i^z - S}{S} - \tfrac{1}{4}\sum_{i,\delta}\frac{\tilde S_i^+ \tilde S_{i+\delta}^- + \tilde S_i^- \tilde S_{i+\delta}^+ + 2(1+\tau)(\tilde S_i^z \tilde S^z}{S^2}$$

where $\tilde f = \mu f/JS\hbar$ is the dimensionless external field. $\tilde S_i^+$, $\tilde S_i^-$ and $\tilde S_i^z$ satisfy the commutation relations

$$[\tilde S_i^{\pm}, \tilde S_j^z] = \mp \tilde S_i^{\pm}\delta_{ij} \qquad (3)$$

$$[\tilde S_i^+, \tilde S_j^-] = 2\tilde S_i^z \delta_{ij} \qquad (4)$$

with $\tilde{\boldsymbol{S}}_i \cdot \tilde{\boldsymbol{S}}_i = S(S+1)$. Then we can introduce the HP (1940) representation for spin operators:

$$\tilde S_i^+ = (2S - a_i^+ a_i)^{1/2} a_i \qquad (5)$$

$$\tilde S_i^- = a_i^+ (2S - a_i^+ a_i)^{1/2} \qquad (6)$$

$$\tilde S_i^z = S - a_i^+ a_i. \qquad (7)$$

a_i and a_i^+ satisfy the Bose commutation relations

$$[a_i, a_j^+] = \delta_{ij} \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{8}$$

If $2S \gg a_i^+ a_i$, we can use the semiclassical approximation

$$\tilde{S}_i^+/S = \sqrt{2}[\varepsilon a_i - \tfrac{1}{4}\varepsilon^3 a_i^+ a_i a_i - \tfrac{1}{32}\varepsilon^5 a_i^+ a_i a_i^+ a_i a_i + \mathrm{O}(\varepsilon^7)] \tag{9}$$

$$\tilde{S}_i^-/S = \sqrt{2}[\varepsilon a_i^+ - \tfrac{1}{4}\varepsilon^3 a_i^+ a_i^+ a_i - \tfrac{1}{32}\varepsilon^5 a_i^+ a_i^+ a_i a_i^+ a_i + \mathrm{O}(\varepsilon^7)] \tag{10}$$

where $\varepsilon = 1/\sqrt{S}$ is a small dimensionless parameter. Then (2) can be written in a power series of ε:

$$\begin{aligned}\tilde{H} = {} & \varepsilon^2\Big(\tilde{f}\sum_i a_i^+ a_i + \tfrac{1}{2}\sum_{i,\delta}[(1+\tau)(a_i^+ a_i + a_{i+\delta}^+ a_{i+\delta}) - a_i a_{i+\delta}^+ - a_{i+\delta} a_i^+]\Big) \\ & + \varepsilon^4 \tfrac{1}{8}\sum_{i,\delta}\{[a_i a_{i+\delta}^+ a_{i+\delta}^+ a_{i+\delta} + a_i^+ a_i a_i a_{i+\delta}^+ + \mathrm{HC}] - 4(1+\tau) a_i^+ a_i a_{i+\delta}^+ a_{i+\delta}\} \\ & + \varepsilon^6 \tfrac{1}{64}\sum_{i,\delta}[a_i a_{i+\delta}^+ a_{i+\delta}^+ a_{i+\delta} a_{i+\delta}^+ a_{i+\delta} + a_i^+ a_i a_i^+ a_i a_i a_{i+\delta}^+ \\ & - 2a_i^+ a_i a_i a_{i+\delta}^+ a_{i+\delta}^+ a_{i+\delta} + \mathrm{HC}] + \mathrm{O}(\varepsilon^8)\end{aligned} \tag{11}$$

where HC represents the corresponding Hermitian conjugate term. The Heisenberg equation of motion for the operator a_j given by

$$\mathrm{i}\hbar\, \partial a_j/\partial t = [a_j, H] \tag{12}$$

can be written in the dimensionless form

$$\mathrm{i}\hbar\, \partial a_j/\partial \tilde{t} = [a_j, \tilde{H}] \tag{13}$$

where $\tilde{t} = \omega_0 t$ is the dimensionless time and $\tilde{\omega}_0 = h\omega_0/JS^2\hbar^2$ (ω_0 is the typical frequency of the waves). Substituting (11) into (13), we have

$$\begin{aligned}\mathrm{i}\tilde{\omega}_0 \partial a_j/\partial \tilde{t} = {} & \varepsilon^2\Big([\tilde{f} + 2(1+\tau)]a_j - \sum_\delta a_{j+\delta}\Big) \\ & + \varepsilon^4 \tfrac{1}{4}\sum_\delta [2a_j^+ a_j a_{j+\delta} + a_{j+\delta}^+ a_{j+\delta}^2 + a_{j+\delta}^+ a_j^2 - 4(1+\tau) a_{j+\delta}^+ a_{j+\delta} a_j] \\ & + \varepsilon^6 \tfrac{1}{32}\sum_\delta (3a_j^+ a_j^+ a_j^2 a_{j+\delta} + 2a_j^+ a_j a_{j+\delta} + a_{j+\delta}^+ a_{j+\delta}^+ a_{j+\delta}^3 + 2a_j^+ a_{j+\delta}^+ a_j^3 \\ & + a_{j+\delta}^+ a_{j+\delta} a_{j+\delta} + a_{j+\delta}^+ a_j a_j \\ & - 4a_{j+\delta}^+ a_j^+ a_{j+\delta} a_{j+\delta} a_j - 2a_{j+\delta}^+ a_{j+\delta}^+ a_j a_j a_{j+\delta}) + \mathrm{O}(\varepsilon^8).\end{aligned} \tag{14}$$

This is the equation of motion for a_j in the semiclassical expansion.

3. Glauber's coherent-state representation and the long-wave approximation

Introducing Glauber's (1963) coherent-state representation

$$|\alpha\rangle = \prod_j |\alpha_j\rangle \qquad a_j|\alpha\rangle = \alpha_j|\alpha\rangle \tag{15}$$

with $\langle\alpha|\alpha\rangle = 1$, equation (14) transforms into

$$\begin{aligned}
\mathrm{i}\tilde{\omega}_0\,\partial\alpha_j/\partial\tilde{t} = {} & \varepsilon^2\Big([\tilde{f} + 2(1+\tau)]\alpha_j - \sum_j \alpha_{j+\delta}\Big) \\
& + \varepsilon^4 \tfrac{1}{4}\sum_\delta [2\alpha_j^*\alpha_j\alpha_{j+\delta} + \alpha_{j+\delta}^*\alpha_{j+\delta}^2 + \alpha_{j+\delta}^*\alpha_j^2 - 4(1+\tau)\alpha_{j+\delta}^*\alpha_{j+\delta}\alpha_j] \\
& + \varepsilon^6 \tfrac{1}{32}\sum_\delta (3\alpha_j^*\alpha_j^*\alpha_j^2\alpha_{j+\delta} + 2\alpha_j^*\alpha_j\alpha_{j+\delta} \\
& + \alpha_{j+\delta}^*\alpha_{j+\delta}^*\alpha_{j+\delta}^3 + 2\alpha_j^*\alpha_{j+\delta}^*\alpha_j^3 + \alpha_{j+\delta}^*\alpha_{j+\delta}^2 + \alpha_{j+\delta}^*\alpha_j^2 \\
& - 4\alpha_{j+\delta}^*\alpha_j^*\alpha_{j+\delta}^2\alpha_j - 2\alpha_{j+\delta}^*\alpha_{j+\delta}^*\alpha_j^2\alpha_{j+\delta}) + \mathrm{O}(\varepsilon^8).
\end{aligned} \tag{16}$$

The next step is to perform the continuum limit

$$\alpha_j(t) \rightarrow \alpha(x, t) \tag{17}$$

$$\begin{aligned}
\alpha_{j\pm1}(t) \rightarrow {} & \alpha(x,t) \pm a\alpha_x + (1/2!)\,a^2\alpha_{xx} \pm (1/3!)a^3\alpha_{xxx} + (1/4!)a^4\alpha_{xxxx} + \ldots \\
& = \alpha \pm \eta\alpha_{\tilde{x}} + (1/2!)\eta^2\alpha_{\tilde{x}\tilde{x}} \pm (1/3!)\eta^3\alpha_{\tilde{x}\tilde{x}\tilde{x}} + (1/4!)\eta^4\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \mathrm{O}(\eta^5)
\end{aligned} \tag{18}$$

$$\sum_i \rightarrow \frac{1}{a}\int \mathrm{d}x = \frac{1}{\eta}\int \mathrm{d}\tilde{x} \tag{19}$$

where a is the lattice constant, $\tilde{x} = x/\lambda_0$ and $\eta = a/\lambda_0$. λ_0 is the typical wavelength of the waves (in the case of soliton excitation, λ_0 will be the soliton width) and η is the dimensionless small parameter used in the long-wave approximation. Then equation (16) becomes

$$\begin{aligned}
\mathrm{i}\tilde{\omega}_0\,\partial\alpha/\partial\tilde{t} = {} & \varepsilon^2[(\tilde{f} + 2\tau)\alpha - \eta^2\alpha_{\tilde{x}\tilde{x}} - \tfrac{1}{12}\eta^4\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \mathrm{O}(\eta^6)] \\
& + \varepsilon^4\{-2\tau|\alpha|^2\alpha + \eta^2[-\alpha|\alpha_{\tilde{x}}|^2 - \tfrac{1}{2}\alpha^2\alpha_{\tilde{x}\tilde{x}}^* + \tfrac{1}{2}\alpha^*(\alpha_{\tilde{x}})^2 \\
& - \tau\alpha(|\alpha|^2)_{\tilde{x}\tilde{x}}] + \mathrm{O}(\eta^4)\} + \varepsilon^6[\tfrac{1}{4}|\alpha|^2\alpha + \mathrm{O}(\eta^2)] + \mathrm{O}(\varepsilon^8).
\end{aligned} \tag{20}$$

All quantities in equation (20) are dimensionless. Here ε and η, which are two small expansion parameters used in the semiclassical approximation and in the long-wave approximation, are written explicitly.

4. The soliton solutions

For a given physical system, ε and η are not independent of each other and can be determined from typical quantities of the system. That is to say, $\eta = g(\varepsilon)$ (g is a function of ε). Theoretically, we could not determine which case is important because different cases correspond to different physical pictures. Only from the experimental conditions and initial exciting conditions, can we estimate which case is suitable. The reason is similar to the theory of non-linear long waves in shallow water (Ablowitz and Segur 1981) which will be discussed in section 5. Here we discuss the soliton solutions in three cases: $\eta = O(\varepsilon)$, $\eta = O(\varepsilon^{3/2})$ and $\eta = O(\varepsilon^2)$.

4.1. The case $\eta = O(\varepsilon)$

We let $\eta = U_1\varepsilon$, with $U_1 = O(1)$. If $\tau \neq 0$, we retain terms up to $O(\varepsilon^4)$ order in equation (20) in order to include the lowest-order non-linear effect. Then we have

$$\mathrm{i}\alpha_t = (\mu f + 2\tau JS_c)\alpha - JS_c a^2\alpha_{xx} - (2JS_c\tau/S)|\alpha|^2\alpha \tag{21}$$

on returning to dimensional variables (we have assumed that $Sh = S_c$). Equation (21) is the non-linear Schrödinger equation obtained firstly by Pushkarov and Pushkarov (1977) in the HP representation. If $\tau > 0$, (21) admits the envelope soliton solution

$$\alpha = (Sa^2/\tau)^{1/2}\nu\ \mathrm{sech}\ \nu\,(x - x_0 + 2JS_c a^2kt)\exp[\mathrm{i}(kx - \omega t - \varphi_0)] \tag{22}$$

with

$$\omega = \mu f + 2\tau JS_c + JS_c a^2(\nu^2 - k^2) \tag{23}$$

where ν, x_0 and φ_0 are integral constants. Here $1/\nu$ (the soliton width) and ω (the vibrating frequency of the soliton) can be thought of as the typical length λ_0 and the typical frequency ω_0 used in equations (13)–(20). The multiple-soliton solutions may be obtained by the inverse scattering transform (Ablowitz and Segur 1981). However, when $\tau = 0$ (the isotropic case), (21) is not valid for describing non-linear excitations of the system (from the solution (22) we can also see that this is true). We must retain the terms in (20) up to $O(\varepsilon^6)$:

$$\begin{aligned}\mathrm{i}\alpha_t = \mu f\alpha - JS_c a^2\alpha_{xx} - \tfrac{1}{12}a^4JS_c\alpha_{xxxx} + (JS_c/4S^2)|\alpha|^2\alpha \\ + (a^2JS_c/S)[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 - \tfrac{1}{2}\alpha^2\alpha^*_{xx}]\end{aligned} \tag{24}$$

on returning to dimensional variables. Since this contains the fourth-order derivative term α_{xxxx}, it is difficult to obtain exact solitary-wave solutions. In the appendix, we use the method of multiple scales (Taniuti and Nishihara 1983) to reduce (24) to the non-linear Schrödinger equation

$$iU_\tau + a^2JS_c(1 - a^2k^2/2)U_{\xi\xi} + [JS_c(a^2k^2 - 1/4S)/S]|U|^2U = 0 \tag{25}$$

with

$$\alpha = \rho\alpha^{(1)} + O(\rho^2) \tag{26}$$

$$\alpha^{(1)} = U(\xi, \tau)\exp[\mathrm{i}(kx - \omega t)] \tag{27}$$

$$\omega = \mu f + a^2JS_ck^2 - JS_ca^4k^4/12 \tag{28}$$

$$\xi = \rho(x - C_gt) \tag{29}$$

$$\tau = \rho^2t \tag{30}$$

where k is the wavenumber and $C_g = \mathrm{d}\omega/\mathrm{d}k$ is the group velocity of the linear wave. ρ is a small parameter denoting the relative amplitude of the waves. The single-soliton solution is

$$\begin{aligned}U = [2a^2S(1 - a^2k^2/2)/(2\tau + a^2k^2 - 1/4S)]^{1/2}\,2\nu \\ \times \mathrm{sech}(2\nu)\,[\xi - \xi_0 + 4\beta a^2JS_c(1 - a^2k^2/2)\tau] \\ \times \exp[-2\mathrm{i}\beta\xi - 4\mathrm{i}(\beta^2 - \nu^2)a^2JS_c(1 - a^2k^2/2)\tau - \mathrm{i}\varphi_0]\end{aligned} \tag{31}$$

where β, ν and ξ_0 are integral constants.

4.2. The case $\eta = O(\varepsilon^{3/2})$

This corresponds to longer waves. We let $\eta = U_2\varepsilon^{3/2}$ with $U_2 = O(1)$. If the terms in (20) are retained up to $O(\varepsilon^7)$, we have

$$\begin{aligned} \mathrm{i}\alpha_t = (\mu f + 2\tau J S_c)\alpha - J S_c a^2 \alpha_{xx} + [(J S_c/4S^2) - 2J S_c \tau/S]|\alpha|^2\alpha \\ + (a^2 J S_c/S)[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 - \tfrac{1}{2}\alpha^2\alpha^*_{xx} - \tau\alpha(|\alpha|^2)_{xx}] \end{aligned} \tag{32}$$

on returning to dimensional variables. If $\tau = 0$, it is the same as the equation obtained by de Azevedo *et al* (1982) (the term $(JS_c/4S^2)|\alpha|^2\alpha$ is missing in their paper). The single-soliton solutions which are identical with those of Tjon and Wright (1977) have been given by them. We refer readers to the paper of de Azevedo *et al* (1982).

4.3. The case $\eta = O(\varepsilon^2)$

This corresponds to even longer waves. We let $\eta = U_3\varepsilon^2$ with $U_3 = O(1)$. The equation including the lowest-order non-linear term is

$$\mathrm{i}\alpha_t = (\mu f + 2J S_c \tau)\alpha - J S_c a^2 \alpha_{xx} - [J S_c(2\tau - 1/4S^2)/S]|\alpha|^2\alpha \tag{33}$$

on returning to dimensional variables. The single-soliton solution is

$$\begin{aligned} \alpha(x,t) = [8S^2a^2/(8S\tau - 1)]^{1/2}\, 2\nu\, \mathrm{sech}[2\nu(x - x_0 + 4\beta J S_c t/S)] \\ \times \exp[-2\mathrm{i}\beta x - 4\mathrm{i}(\beta^2 - \nu^2) J S_c t/S - \mathrm{i}(\mu f + 2\tau J S_c)t - \mathrm{i}\varphi_0] \end{aligned} \tag{34}$$

where β, ν, x_0 and φ_0 are integral constants. Equation (33) admits the envelope soliton solution (34) only when $8S\tau > 1$. If $8S\tau = 1$, (33) is not valid for describing non-linear effects of the system since the non-linear term in (33) vanishes. In this case, we must include the higher-order terms in (20).

5. Discussion and summary

We have investigated soliton excitations in the 1D Heisenberg chain. From the above we can see that the relative ratio of ε to η is important for determining the modified terms of the equation of motion in the semiclassical approximation and the long-wave approximation. This is an example of Kruskal's (1963) 'principle of maximal balance', which states that in a perturbation expansion involving two or more small parameters a scaling which reduces the problem as little as possible is of interest.

Škrinjar *et al* discussed the non-linear excitations in the Heisenberg chain using the model Hamiltonian (1). In their approach, the continuum limit (long-wave approximation) was used by keeping the derivative terms to the order $a^2\,\partial^2/\partial x^2$, and in the semiclassical approximation they retained terms up to products of six Bose operators (Škrinjar *et al* 1987) or the complete series of the expansions of the HP transformation (Škrinjar *et al* 1989). Since a comparison between the orders of the semiclassical approximation and the long-wave approximation was not taken into account, there is a little confusion in their calculations.

In Glauber's coherent-state representation, the Hamiltonian (1) can be written as

$$\langle\alpha|(H-H_0)|\alpha\rangle = JS^2\hbar^2\langle\alpha|\tilde{H}|\alpha\rangle \tag{35}$$

$$\langle\alpha|\tilde{H}|\alpha\rangle = \frac{1}{a}\int \mathrm{d}x\, \mathscr{H}(x,t) = \frac{1}{\eta}\int \mathrm{d}\tilde{x}\, \mathscr{H}(\tilde{x},t) \tag{36}$$

$$\begin{aligned}\mathscr{H} = {} & \varepsilon^2\{(\tilde{f}+2\tau)|\alpha|^2 + \tfrac{1}{2}\eta^2[(1+\tau)(|\alpha|^2)_{\tilde{x}\tilde{x}} - \alpha\alpha^*_{\tilde{x}\tilde{x}} - \alpha^*\alpha_{\tilde{x}\tilde{x}}] \\ & + \tfrac{1}{24}\eta^4[(1+\tau)(|\alpha|^2)_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} - \alpha\alpha^*_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} - \alpha^*\alpha_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}] + \mathrm{O}(\eta^6)\} \\ & + \varepsilon^4\{-\tau|\alpha|^4 + \tfrac{1}{8}\eta^2[\alpha^*(|\alpha|^2\alpha)_{\tilde{x}\tilde{x}} + |\alpha|^2\alpha^*\alpha_{\tilde{x}\tilde{x}} + \alpha(|\alpha|^2\alpha^*)_{\tilde{x}\tilde{x}} + |\alpha|^2\alpha\alpha^*_{\tilde{x}\tilde{x}} \\ & - 4(1+\tau)|\alpha|^2(|\alpha|^2)_{\tilde{x}\tilde{x}} + \mathrm{O}(\eta^4)\} \\ & + \varepsilon^6[\tfrac{1}{8}|\alpha|^4 + \mathrm{O}(\eta^2)] + \mathrm{O}(\varepsilon^8)\end{aligned} \tag{37}$$

where ε, η and $\tilde{x}$ are defined as above. In our framework, the discussions of Škrinjar *et al* (1987, 1989) can be thought of as taking the Hamiltonian to be one in which all $\mathrm{O}(\eta^4)$ terms in (37) are neglected and then $\eta = \mathrm{O}(1)$ is employed.

We now use the development of the theory of non-linear long waves in shallow water to support our approach given above. In the dimensionless form of the Euler equations, there are two small parameters

$$\varepsilon = A/h \qquad \mu = kh \tag{38}$$

where A is the typical amplitude of the waves, h is the static depth of water, $k = 2\pi/\lambda$, λ is a typical wavelength, ε is the small parameter used in the weak-amplitude approximation and μ is the small parameter used in the long-wave approximation (Ablowitz and Segur 1981). Historically, there had been two kinds of non-linear shallow-water wave theories: one was the Airy theory which corresponds to

$$\varepsilon = \mathrm{O}(1) \qquad \mu \ll 1 \tag{39}$$

and the other was the Boussinesq–Korteweg–de Vries theory which takes

$$\varepsilon = \mathrm{O}(\mu^2) \ll 1. \tag{40}$$

The two theories gave quite different results. This paradox was not solved until 1953 (Ursell 1953). Ursell pointed out that the relative ratio of ε to μ is very important when making the perturbation expansion of the Euler equations. Different ratios correspond to different physical conditions. The number

$$\mathrm{Ur} = \varepsilon/\mu^2 = A\lambda^2/(2\pi)^3h^3 \tag{41}$$

is now called the Ursell number.

It should be pointed out that in our approach only the parameters $\varepsilon = 1/\sqrt{S}$ and $\eta = a/\lambda_0$ are assumed to be small quantities. The other dimensionless parameters τ, $\tilde{f}$ and σ ($\sigma = |\alpha_j|_{\max}$, where α_j are the coherent amplitudes) have been thought to be of O(1). From the mathematical point of view, the semiclassical approximation and small-amplitude approximation are not the same, because S is the characteristic quantity of the system whereas σ is determined by the external exciting conditions. We can also let

ε, η and σ be small parameters and make another perturbation expansion. This can be done by defining $a_i = \sigma\tilde{a}_i$ in the Hamiltonian (11). Then (11) becomes

$$\tilde{H} = (\varepsilon\sigma)^2 \Big(\tilde{f}\sum_i \tilde{a}_i^+\tilde{a}_i + \tfrac{1}{2}\sum_{i,\delta}[(1+\tau)(\tilde{a}_i^+\tilde{a}_i + \tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta} - \tilde{a}_i\tilde{a}_{i+\delta}^+ - \tilde{a}_{i+\delta}\tilde{a}_i^+]\Big)$$
$$+ (\varepsilon\sigma)^4\tfrac{1}{8}\sum_{i,\delta}[(\tilde{a}_i\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta} + \tilde{a}_i^+\tilde{a}_i\tilde{a}_i\tilde{a}_{i+\delta}^+ + \text{HC})$$
$$- 4(1+\tau)\tilde{a}_i^+\tilde{a}_i\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta}] + (\varepsilon\sigma)^6\tfrac{1}{64}\sum_{i,\delta}$$
$$\times[\tilde{a}_i\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta}\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta} + \tilde{a}_i^+\tilde{a}_i\tilde{a}_i^+\tilde{a}_i\tilde{a}_i\tilde{a}_{i+\delta}^+$$
$$- 2\tilde{a}_i^+\tilde{a}_i\tilde{a}_i\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta}^+\tilde{a}_{i+\delta} + \text{HC}] + \mathrm{O}(\varepsilon\sigma)^8. \tag{42}$$

Because ε and σ always appear within the combination $\varepsilon\sigma$, we can define $\varepsilon' = \varepsilon\sigma$ and take ε' and η as the expansion parameters. This procedure is equivalent to the following two kinds of expansion:

$$\varepsilon \ll 1 \qquad \eta \ll 1 \qquad \tilde{f} = \mathrm{O}(1) \qquad \tau = \mathrm{O}(1) \qquad \sigma = \mathrm{O}(1) \tag{43}$$

$$\sigma \ll 1 \qquad \eta \ll 1 \qquad \tilde{f} = \mathrm{O}(1) \qquad \tau = \mathrm{O}(1) \qquad \varepsilon = \mathrm{O}(1). \tag{44}$$

Equation (43) corresponds to the semiclassical approximation and the long-wave approximation used in this paper. Equation (44) corresponds to the small-amplitude approximation and the long-wave approximation. It is easy to see that the equation which $\tilde{a}_j$ satisfies will still be (14) with ε and a_j being substituted by ε' and $\tilde{a}_j$. The reason is that the Hamiltonian (42) has the same form as (11).

In summary, we have studied soliton excitations in a 1D Heisenberg ferromagnet using the HP transformation and Glauber's coherent-state representation. Writing the Hamiltonian in the dimensionless form, the equation of motion is transformed into the power series of ε and η in the semiclassical approximation and in the long-wave approximation. We have shown that the relative ratio of ε to η is important for the determination of the modified Schrödinger equation. The soliton solutions are given in three cases which correspond to different physical conditions of the system. The results obtained by Pushkarov and Pushkarov, de Azevedo *et al* and Škrinjar *et al* may be included in our approach. The idea is generated from the theory of non-linear long waves in shallow water. The physical application of the approach given here will be discussed in a future publication.

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Appendix

Here we use the method of multiple scales (Taniuti and Nishihara 1983) to solve equation (24):

$$\mathrm{i}\alpha_t + A\alpha + B\alpha_{xx} + C\alpha_{xxxx} + D|\alpha|^2\alpha + E[-\alpha|\alpha_x|^2 + \tfrac{1}{2}\alpha^*(\alpha_x)^2 - \tfrac{1}{2}\alpha^2\alpha^*_{xx}] = 0 \qquad \text{(A1)}$$

where

$$A = -\mu f \qquad B = JS_\mathrm{c}a^2 \qquad C = JS_\mathrm{c}a^4/2$$
$$D = -JS_\mathrm{c}/4S^2 \qquad E = -JS_\mathrm{c}a^2/S. \qquad \text{(A2)}$$

For weak non-linear waves with dispersion, we can introduce the slow variables

$$\xi = \rho(x - C_\mathrm{g}t) \qquad \text{(A3)}$$

$$\tau = \rho^2 t \qquad \text{(A4)}$$

and asymptotic expansion

$$\alpha = \rho\alpha^{(1)} + \rho^2\alpha^{(2)} + \rho^3\alpha^{(3)} + \ldots \qquad \text{(A5)}$$

where ρ is a small parameter denoting the relative amplitude of the waves. $C_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k$ is the group velocity of the linear waves. From (A3) and A(4) we have the derivative expansions (Nayfeh 1973)

$$\partial/\partial x = \partial/\partial x + \rho\,\partial/\partial\xi \qquad \text{(A6)}$$

$$\partial/\partial t = \partial/\partial t - \rho C_\mathrm{g}\partial/\partial\xi + \rho^2\partial/\partial\tau. \qquad \text{(A7)}$$

Substituting (A5), (A6) and (A7) into (A1) and equating the coefficients of like powers of ρ, we obtain

$$\mathrm{i}\alpha^{(j)}_t + A\alpha^{(j)} + B\alpha^{(j)}_{xx} + C\alpha^{(j)}_{xxxx} = \gamma^{(j)} \qquad j = 1, 2, 3, \ldots \qquad \text{(A8)}$$

$$\gamma^{(1)} = 0 \qquad \text{(A9)}$$

$$\gamma^{(2)} = \mathrm{i}C_\mathrm{g}\alpha^{(1)}_\xi - 2B\alpha^{(1)}_{x\xi} - 4\alpha^{(1)}_{xxx\xi} \qquad \text{(A10)}$$

$$\begin{aligned}\gamma^{(3)} = {} & \mathrm{i}C_\mathrm{g}\alpha^{(2)}_\xi - \mathrm{i}\alpha^{(1)}_\tau - B(2\alpha^{(2)}_{x\xi} + \alpha^{(1)}_{\xi\xi}) - C(4\alpha^{(2)}_{xxx\xi} + 6\alpha^{(1)}_{xx\xi\xi}) \\ & - D|\alpha^{(1)}|^2\alpha^{(1)} - E[\tfrac{1}{2}|\alpha^{(1)}|^2\alpha^{(1)}_{xx} + \tfrac{1}{4}(|\alpha^{(1)}|^2\alpha^{(1)})_{xx} \\ & + \tfrac{1}{4}\alpha^{2(1)}\alpha^{*(1)}_{xx} - \alpha^{(1)}(|\alpha^{(1)}|^2)_{xx}] \\ & \vdots \end{aligned} \qquad \text{(A11)}$$

Then we can solve these equations step by step.

For $j = 1$ we obtain the linear approximation solution

$$\alpha^{(1)} = U(\xi, \tau)\exp[\mathrm{i}(kx - \omega t)] \qquad \text{(A12)}$$

$$\omega = \mu f + a^2JS_\mathrm{c}k^2 - JS_\mathrm{c}a^4k^4/12 \qquad \text{(A13)}$$

where $U(\xi, \tau)$ is an undetermined function. When $j = 2$, we have the second-order approximation equation. It is easy to get

$$\alpha^{(2)} = V(\xi, \tau)\exp[\mathrm{i}(kx - \omega t)] \qquad \text{(A14)}$$

$$C_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k = 2JS_\mathrm{c}a^2k - JS_\mathrm{c}a^4k^3/3 \qquad \text{(A15)}$$

where $v(\xi, \tau)$ is another undetermined function. The third-order ($j = 3$) approximation equation is

$$\mathrm{i}\alpha_t^{(3)} + A\alpha^{(3)} + B\alpha_{xx}^{(3)} + C\alpha_{xxxx}^{(3)} = -[\mathrm{i}U_\tau + (B - 6Ck^2)U_{\xi\xi} + (D - 2Ek^2)|U|^2 U] \times \exp[\mathrm{i}(kx - \omega t)] + \text{higher-order harmonic term.} \tag{A16}$$

In order to eliminate the secular terms in $\alpha^{(3)}$, we must let

$$\mathrm{i}U_\tau + (B - 6Ck^2)U_{\xi\xi} + (D - 2Ek^2)|U|^2 U = 0. \tag{A17}$$

This is the non-linear Schrödinger equation given in (25).

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